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ON THE UNDULAR HYDRAULIC JUMP AND THE UNDULAR SURGE

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ABSTRACT

Whereas the undular hydraulic jump and the undular hydraulic surge may be considered similar, recent experimental and computational knowledge reveal a marked difference between the two: The main reasons for the modifications are based on their significantly altered states of boundary layer development, and the extremely small roughness effects as to the surges. This historical work deals with the advances made in the description of undular flows in hydraulic practice also pointing at strategies for the solution of undular flows under real fluid flow conditions. Reasons for the relevance of undular flows in hydraulic engineering are also discussed. The work is illustrated with available plots and photos from the engineering literature.

Keywords: History of hydraulics, Open channel hydraulics, Undular hydraulic jump, Undular surge

1 INTRODUCTION

A hydraulic jump corresponds to the transition of super- to subcritical flows, associated with highly turbulent flow properties such as pressure and velocity, air entrainment and return flow at the free surface from the tailwater toward the toe of the jump. In general, two types of hydraulic jumps are distinguished, namely the direct hydraulic jump, and the undular hydraulic jump. The first is relevant in hydraulic engineering, given its capacity to significantly dissipate hydraulic energy so that no high velocity and wave action occurs in the tailwater beyond the stilling basin. These basins are a standard design mainly for high-head power plants, given that velocities at the dam base may easily be higher than 50 m/s. Typically, stilling basins are provided for inflow velocities of up to a maximum of 20 m/s, to exclude problems with cavitation damage and wave generation in the tailwater, along with the formation of spray as a nuisance close to the switchyard of power plants.

In contrast, the undular hydraulic jump is a transitional phenomenon occurring at approach flow (subscript 1) Froude numbers slightly above F_1=1. In the rectangular channel, to be considered exclusively here, the Froude number is defined as $F=U/(gh)^{1/2}$, with U as the cross-sectional average flow velocity, g as the gravity acceleration, and h as the flow depth. Four types of undular hydraulic jumps were proposed by Reinauer and Hager (1995) with $F^A \cong 1.20$, $F^B \cong 1.28$, $F^C \cong 1.36$, and $F^D \cong 1.6$:

- Type A if $1<F\leq F^A$, involving nearly 2D surface undulations, without any shocks and surface rollers
- Type B if $F^A \leq F<F^B$, along with the development of lateral shocks from the first wave crest. The intersecting waves continue into the tailwater, generating a 3D flow pattern
- Type C if $F^B \leq F<F^C$, for which a surface roller occurs at the intersection of the two first shocks. This roller is small and limited to the first wave length only
- Type D if $F^C \leq F<F^D$, under which the roller breaks thereby choking the surface current along with air entrainment. If $F_1 > F^D$, the undular surface pattern is lost generating the weak direct hydraulic jump.

The typical flow features of these jumps are detailed and illustrated by Reinauer and Hager (1995). They also point at possible scale effects due to viscosity and surface tension if the approach flow depth h_1<50 mm. These configurations were excluded, to result in a purely gravitational flow with the Froude number as dominant factor.

The question is: What is the significance of undular hydraulic jumps in hydraulic engineering, or why should we be interested into these flows? As stated, only direct hydraulic jumps are employed in stilling basins, with F_1>3 to become effective as a dissipation structure. So why do we look for the undular hydraulic jump? In nature, the direct hydraulic jump is hardly visible, given the massive roughness pattern due to the presence of large boulders in steep slopes. In turn, the undular hydraulic jump may be noted, not as previously classified because of the more complex environment in terms of channel geometry and roughness pattern, but as a complex phenomenon pointing thereby also at the beauties of these turbulent and wavy flows. If the available head upstream from its toe is too small to generate a direct jump, then the flow may be considerably disturbed by the presence of free surface undulations, generating waves along riverbanks or velocity concentrations in the tailwater, which are not a standard hydraulic design despite the flow beauties. Undular jumps should thus be avoided in hydraulic structures because of their instability pattern and their enormous length, creating problems in the transition from concrete-made hydraulic conveyances to the natural river in its tailwater. Therefore, the main features of undular hydraulic jumps should be known to the hydraulic designer. The purpose of this paper is to provide a background, along with an outline into the future treatment of these complex flows.

The reader is remembered that up to 1986 the undular (steady) hydraulic jump was believed to be the equivalent to the undular (unsteady) bore advancing over still water, simply transforming the latter unsteady motion into a steady flow by resorting to a Galilean transformation of coordinates (see e.g. Jones 1964). Montes (1986) aptly highlighted the error given the different boundary layer behavior, in addition to the 3D features of the undular jump as compared to the almost perfect 2D structure of the undular bore. As this note is historical, however, there is no attempt below to 'rectify' those works relating to undular bores and labeled as 'undular jumps' by their authors. The paper emphasis is on the undular hydraulic jump, given the limited space, so the undular bore is only marginally quoted.

2 STUDIES UP TO 1986

The history of the hydraulic jump started exactly 200 years ago, given that Giorgio Bidone (1781-1839) conducted the first experiments in 1819, which were continued in 1824. His values of F_1 were so small that mainly undular jumps were generated. As a byproduct, Italians still refer to the 'salto di Bidone' in their language. The first mechanical analysis was erroneously made by Jean-Baptiste Bélanger (1790-1874) in 1828, given that he based computations onto the energy conservation principle. His 1838 work then involved the momentum conservation principle, yet the differences between the two resulting expressions were small because of the values of F_1 close to unity (Rouse and Ince 1957). The first definite observations on undular hydraulic jumps were conducted by the best experimenter of the 19th century, Henry Bazin (1829-1917). As a by-product of his 1865 Report on Hydraulic researches, he considered also these flows, describing them with the details for which he is known. Despite Henry Darcy (1803-1858) initiated this research program, he was no more alive at the time these observations were taken. Despite, the total work is known as that of Darcy and Bazin (1865).

The first laboratory experiments on undular hydraulic jumps were conducted by Josef Einwachter (1899-1955). His biography is detailed by Hager (2001) relating to his works on stilling basins. Einwachter (1935) refers to two of his previous works on the direct hydraulic jump. His work also includes limited observations on the undular hydraulic jump. Note that all his approach flow depths were considerably below the previously stated limit value of h_1=50 mm. His finding, reproduced in Fig. 1, relates to a particular aspect of the undular hydraulic jump, namely the formation of a (central) bottom roller below the wave crests due to 'boundary layer separation from the channel bottom'. Einwachter refers to an experiment in which F_1=1.5, for which the first relative wave height was h_w/h_1=2. Increasing the value of F_1 results in wave breaking, as above explained for Jump type C.

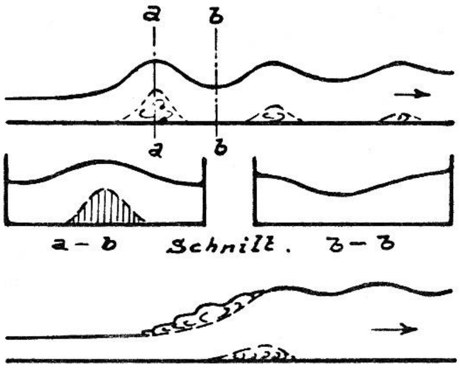


Figure 1. Undular hydraulic jump with streamwise section, 'a' and 'b' indicating wave crest and trough sections (top), Sections a-a and b-b (center), and streamwise section for higher F_1 (bottom) (Einwachter 1935)

Lauffer (1935) was the second experimenting on undular jumps. Based on the standard formulae for the direct hydraulic jump based on Bélanger, the sequent depth ratio and the energy loss compare well with his test data. However, for undular hydraulic jumps, the first crest height is significantly smaller, whereas the energy loss is larger. The effects of the weight component in the flow direction and the boundary roughness were then considered, resulting in a maximum deviation of 8% between the standard theory and the tests. Lauffer also noted a practically constant flow depth along the channel walls, whereas undulations are confined to the center channel portion. Figure 2 shows a qualitative description. These results again were based on too small approach flow depths h_1.

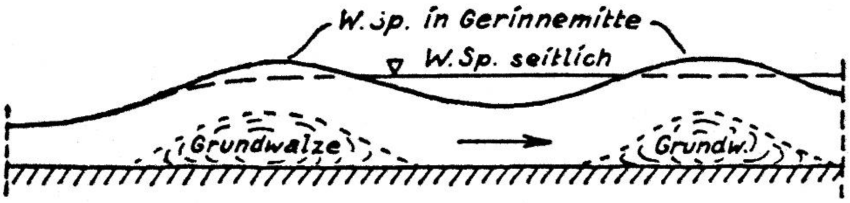


Figure 2. Undular hydraulic jump as plotted by Lauffer (1935) with 'Grundwalze' as bottom roller, 'W.sp. in Gerinnemitte' as flow surface along channel axis, and 'W.sp. seitlich' as lateral flow surface

The works of Favre (1935), Lemoine (1948), Benjamin and Lighthill (1954), and Binnie and Orkney (1955) are not further detailed due to space limitations. Fawer (1937) conducted a systematic study on undular jumps, measuring the free surface profile, and the velocity and pressure distributions. One of his main experimental findings was the demonstration of the complex turbulent velocity profile of the undular jump, influenced by both viscous and streamline curvature effects.

Iwasa (1955) first states the necessary conditions to approximately tackle with the undular hydraulic jump, namely (1) vertical flow accelerations have to be accounted for, (2) frictional effects are small, and (3) the effect of bottom slope is nearly compensated for by wall friction. Considerations are limited to $F_1^2<1+2^{1/2}=1.55^2$, i.e. close to the upper limit of Type B jumps. Iwasa then introduces the three mathematical relations requiring that the discharge and streamwise momentum do not change in the streamwise direction, plus the momentum relation including free surface streamline inclination and curvature effects. An immediate result is the equation of the solitary wave, whose crest height equals $h_m/h_1=F_1^2$. The solution of the undular jump is then made up by a starting solitary wave beyond its crest, from where a steady cnoidal wave is attached to the former profile. The two profiles are connected at the so-called transcritical point. The above limit given for the appearance of non-breaking undular jumps is then established based on the breaking criterion of solitary waves. Experiments were conducted to verify the theoretical results. It was found that the tailwater height indeed follows the standard sequent depth ratio, irrespective whether undular, breaking or direct jumps occur. As to the maximum wave height, it was found to be identical to the solitary wave height up to $F_1\cong1.6$, whereas it is much smaller for breaking waves due to considerable energy dissipation (Fig. 3). These two findings founded the reputation of the great Japanese hydraulician Iwasa.

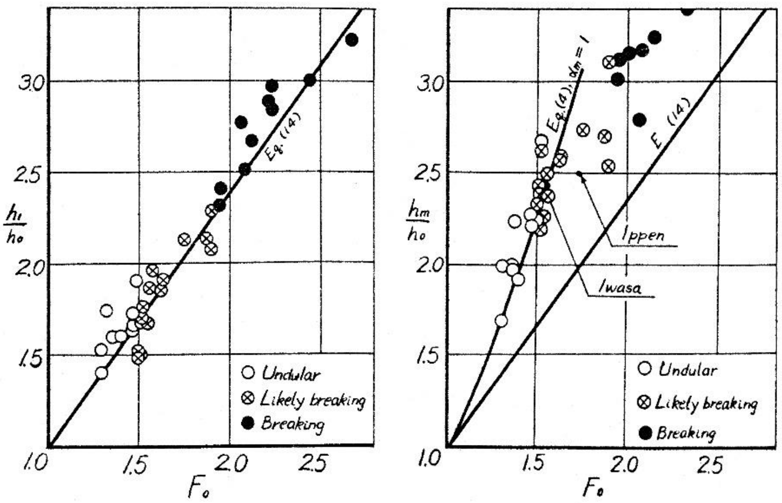


Figure 3. Undular hydraulic jump with sequent depth ratio (left), relative maximum jump height (right), both versus approach flow Froude number (Iwasa 1955)

Marchi (1963) considers steady curvilinear potential flow in a rectangular prismatic channel. Based both on the conservation equations of energy and momentum, he determines expressions for flows with slightly curved streamlines, thereby invoking the terms $(dh/dx)^2$ and $h(d^2h/dx^2)$, as proposed by Boussinesq (1877), with x as the streamwise coordinate and $h=h(x)$ the free surface profile measured vertically from the horizontal bottom. Two cases are distinguished: (I) Uniform upstream flow adjusting to gradually varied tailwater flow across an undular jump for bottom slope larger than the critical slope ($S_o>S_c$); (II) Gradually varied upstream flow adjusting to uniform tailwater flow ($S_o<S_c$). As previously, it is stated that the height of the first wave crest is identical to that of the corresponding solitary wave.

Jones (1964) may be considered the first having added to the topic in terms of test data. Given that his computations are based on shallow water considerations excluding streamline curvature, these will be excluded. The experiments were conducted in a rectangular channel 0.45 m wide and 0.60 m deep, and a bottom slope of 0.20%. The approach flow depths ranged from h_1=0.06 m to 0.21 m, thereby excluding the mentioned scale effects. As to the propagation speed c, the theoretical result $C=c/(gh_1)^{1/2}$ versus the relative wave height $A=a/h_1$ is in agreement with the observations, namely

$$C=\left[1+A\right]^{1/2}. \qquad [1]$$

With $(a'+h_1)$ as the tailwater depth, and $(a+h_1)$ as the maximum height of the undular jump, Jones confirms experimentally with $A'=a'/h_1$ and $A=a/h_1$ the relation

$$A = A'\left[1 + \frac{1}{2}A'\right]. \qquad [2]$$

King, the first discusser of Jones' paper, states that an estimate of the peak height is simply A=(3/2)A'. Rouse, the second discusser, thinks that the undular hydraulic jump resembles more a cnoidal than a solitary wave. He congratulates the author for having published his results from a PhD thesis submitted in 1941. Engel, the third discusser, questions Jones' approach given that viscous effects are neglected. He thereby overlooks the range of approach flow depths h_1 well below the limit of 0.05 m, as quoted above. In closing, Jones states that the problem presented is far from being solved. Much attention should be directed to the accurate reading of lengths and velocities, and "that the analytical researches have outpaced the experimental investigations, but if these latter are to produce useful results, the errors of measurement must be kept under close control."

Peregrine (1966) considers undular bores as a transition between different uniform flows. If the transition between still water and deeper water has initially a small slope, the latter will steepen and form a bore. Experiments of Favre (1935) indicate that undular bores form if $A \leq 0.28$. For $0.28 < A < 0.75$, there are still undulations but the first wave at least breaks. Given the overall shallowness, Peregrine stipulates that these flows can be tackled with the Shallow Water Equations. The two relevant parameters then are the relative wave amplitude versus the approach flow depth A, and the flow depth versus the wavelength. Peregrine conducted numerical experiments based on the Boussinesq equations (Castro-Orgaz and Hager 2017) using finite difference approximations. A comparison of the results based on the Boussinesq and the Airy equations reveals differences mainly in the steep wave regions, given that Airy accounted for hydrostatic pressure. Figure 4 shows a plot of Peregrine (1966) for the maximum wave height η_{max}/η_0 versus $\eta_0 = z/h_0$ as the displacement of the water surface from the original flow depth, here h_0. Except for the data of Sandover and Zienkiewicz (1957), the data portray a clear trend by increasing almost linearly for $0 \leq \eta_0 \leq 0.3$, from where they decay due to wave breaking.

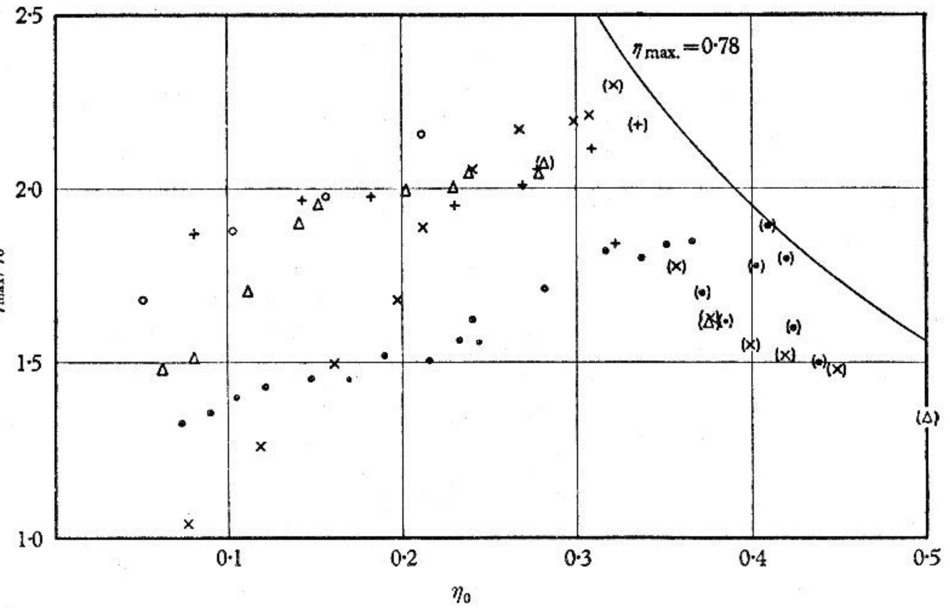


Figure 4. Maximum amplitude of undular bores η_{max}/η_0 versus η_0 along with experimental data. (•) data of Sandover and Zienkiewicz (1957) deviating from main trend, ( ) data indicating wave breaking

Meyer (1967) considers the undular surge by a computational approach. In contrast to previous studies, he thinks that dissipation cannot be disregarded, because this is unconvincing for lack of any physically founded mechanism. His considerations include only waves propagating into fluid at rest, and of small amplitude. His analysis is based on the conservation of mass and momentum. Three different wave types are investigated, namely the Airy, the Jeffreys, and the Korteweg-de Vries (KdV) waves. Airy waves are excluded because they are unable to describe the phenomena considered. The same statement applies to Jeffreys waves, so the only option for describing the undular surge is based on KdV waves. The studies of Preissmann and Cunge (1967) and Benet and Cunge (1971) are not further discussed because they relate to the undular jump in the trapezoidal channel.

Holtorff (1967) considers steady undular jumps based on the Boussinesq equation. He finds that these are physically possible only for $h_2/h_1 < 1.85$, or $F_1 < 1.62$. Based on previous data, his results include also relations between the extreme wave heights, the first relative wave length and the sequent depth ratio. Wave breaking occurs if $F_1 > 1.28$ for the steady arrangement, whereas this limit may be much higher at $F_1 > 1.85$, based on Ippen and Harleman (1956) for undular surges. The main reason for the discrepancy is stated to be the difference in the velocity profiles. A second reason may be the effect of the hydraulic radius, or the relative channel width with respect to the approach flow depth. Tursunov (1969) presents an alternative approach not to be reviewed here, and particularly adds numerous citations mainly originating from the former USSR.

Abbott and Rodenhuis (1972) started their computational analyses from the Boussinesq equations, which were transformed into difference equations. The computational initiation was at a region of constant state from where numerical results as shown in Fig. 5 resulted. These plots served as basis to determine wave heights and wave lengths. The wave length was found to vary strongly with the computational and the temporal steps, because their extreme sensitivities to small perturbations in energy or momentum flux. The results obtained still nearly agree with the experimental data of Favre (1935). In discussing these results, Peregrine presents various sets of the 'Boussinesq equations', stating that different solutions result, depending on the problem considered.

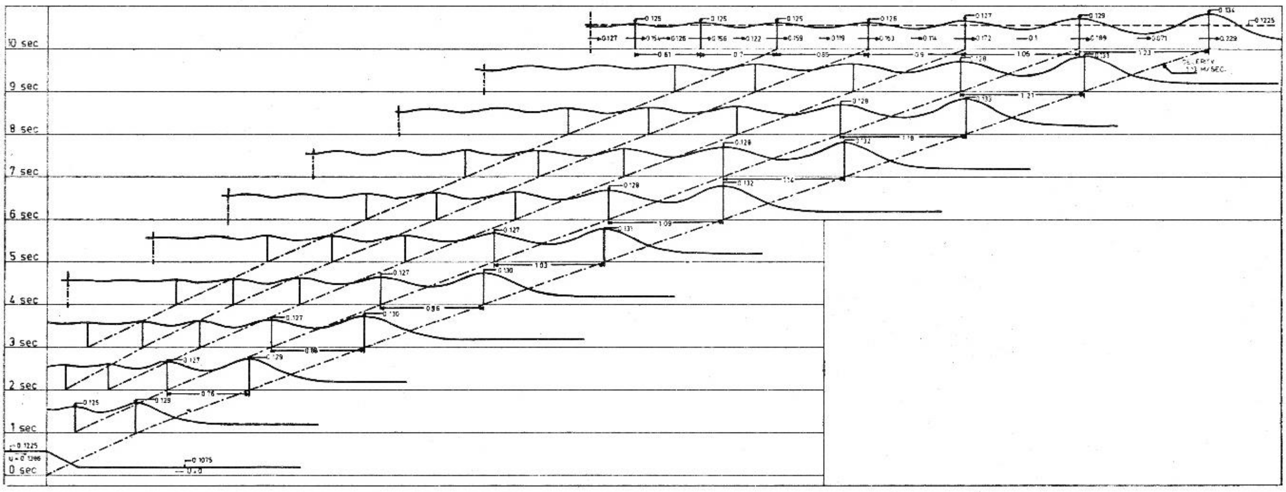


Figure 5. Numerical simulation of undular surge propagating along channel, starting with an initial slope of 0.05 (lower left) and propagating over 10 s (Abbott and Rodenhuis 1972)

Marchi (1974) describes the surface profile of the undular hydraulic jump by usage of the Boussinesq equations expressing conservation of energy head and streamwise momentum. Assuming, as usually, that the friction slope is compensated for by the bottom slope, the resulting equation may be integrated once, following Boussinesq (1877). Assuming further uniform flow conditions at the upstream section then allows for the prediction of the free surface profile. Figure 6a shows the result and compares the upstream portion with the profile of the corresponding solitary wave, indicating excellent agreement of the two. Figure 6b shows the same undular jump but with the presence of a small rod of 2.8 mm diameter positioned at x_o=90 mm from the origin. The resulting free surface profile is now completely modified, indicating the significant effect of a minute change of the boundary arrangement on the resulting profile, as also discussed by Benjamin and Lighthill (1954). This finding also sheds light on the effect of a loose boundary geometry, as occurs with the presence of bed load. Computations then are much more involved, and the effect of the sediment on the flow features becomes complex. Few definite results are currently available on this sediment-water two-phase flow for near-critical flows. Marchi (1974) by the way also compared the velocity profiles with and without rod presence, finding small differences between the two close to the free surface, but marked differences close to the bottom due to the changes of the boundary layer thickness in the wake zone of the rod.

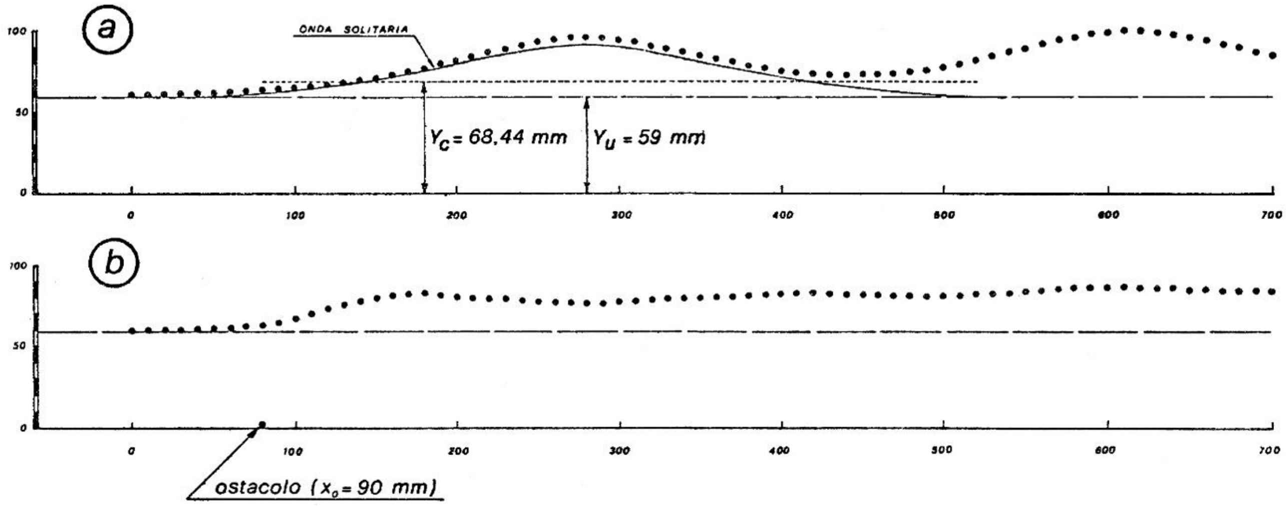
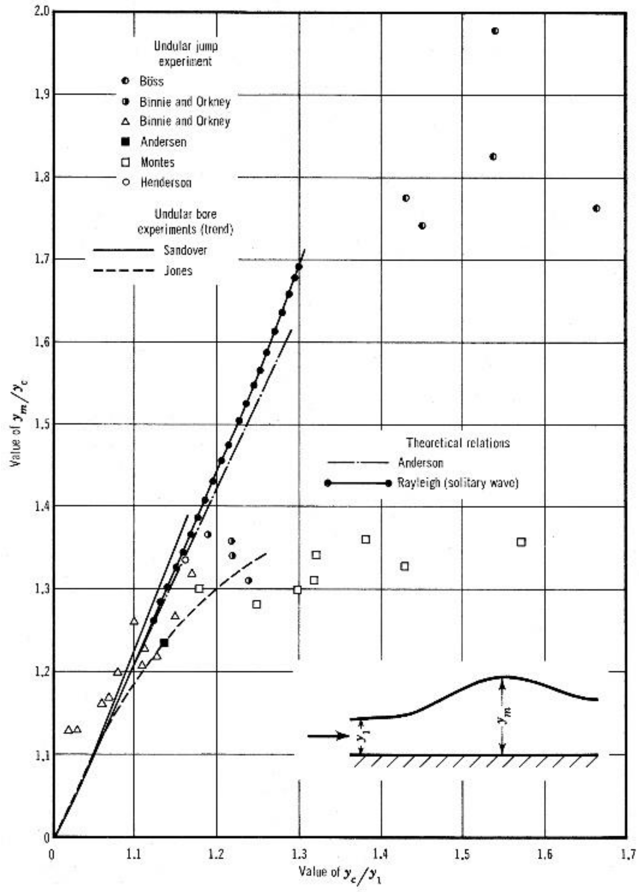


Figure 6. (•) Computed free surface profile and (–) solitary wave profile for horizontal bed and (a) rod absence, (b) rod presence (Marchi 1974)

Mandrup Andersen (1978) proposed an alternative computational approach for the undular hydraulic jump. Based again on the standard Boussinesq equation, the free surface profile is integrated by a step-by-step procedure. By linearizing, an expression for the wavelength L is obtained in terms of the critical depth $h_c=(Q^2/gb^2)^{1/3}$ with $\lambda=L/h_c$ and $y_2=h_2/h_c$ as

$$\lambda = \frac{2\pi}{\left[3(y_2 - \frac{1}{y_2^2})\right]^{1/2}} \quad [3]$$

By discussing the results of this paper, Montes correctly states that the role of energy dissipation on the flow features of undular jumps was by then not correctly understood. The boundary layer along the channel bottom is fully turbulent, as opposed to the laminar character under an advancing undular bore. Montes continues that non-negligible energy dissipation mainly along the leading jump portion was experimentally found, indicating some 5% energy loss from the jump start to the first wave crest. In turn, practically no losses were measured between the first crest and the first wave trough. Given the length of the entire jump, viscous effects must be accounted for to explain the decreasing height of the wave amplitudes from the first wave crest to the ultimate nearly horizontal free surface. Figure 7a shows a plot of Montes in which the relative first wave height y_m is plotted against the critical depth $y_c=[Q^2/(gb^2)]^{1/3}$, with Q as discharge, b as channel width, g as the gravity acceleration and $y_c/y_1=F_1^{2/3}$. Note that for $y_c/y_1<1.1$, both undular bores and jumps follow the same trend, whereas the data split as F_1 increases, resulting in a plateau value of $y_m/y_c=1.35$ for jumps. Mandrup Andersen, in his Closure to the paper, explains that the effect of energy loss across an undular hydraulic jump is not at all clear, and that Montes' view is not a final result. Mandrup Andersen continues that the purpose of his paper was only to describe the flow development up to the first wave crest, excluding the rear wave portion because of the uncertainties relating to the energy losses. He presents a final plot (Fig. 7b) in which the data shown by Montes are subdivided into regions of the solitary wave, the undular jump, and the ordinary (direct) hydraulic jump. As noted, the data of Montes deviate from the general trend.



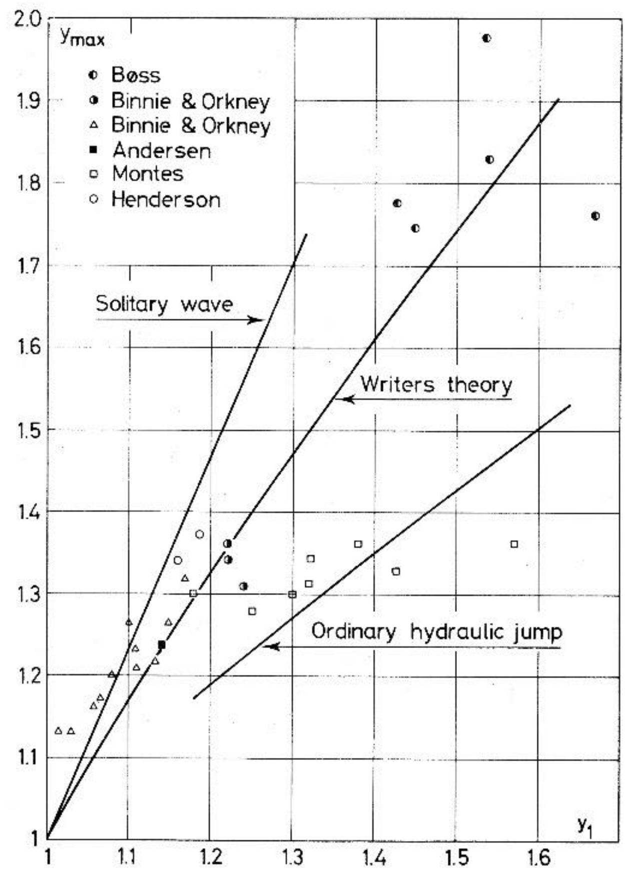


Figure 7. (Left) First crest height of undular jump versus F_1, with $y_c/y_1=F_1^{2/3}$. Comparison between undular jump and undular bore experiments based on Discussion of Mandrup Andersen (1978). (Right) Maximum undular jump height versus approach flow depth as compared with expressions relating to the solitary wave and the ordinary hydraulic jump (Closure of Mandrup Andersen)

3 STUDIES FROM 1986 ONWARDS

Montes (1986) marked the beginning of a new period in the study of the undular hydraulic jump. His results relate to: (i) demolition of the undular surge analogy [Fig. 8 (left)], (ii) portray of the 3D features of the free surface profile linked to the development of lateral boundary layers due to the adverse pressure gradient [Fig. 8 (right)], and (iii) inclusion of real fluid flow effects in the approximate 1D Boussinesq equation.

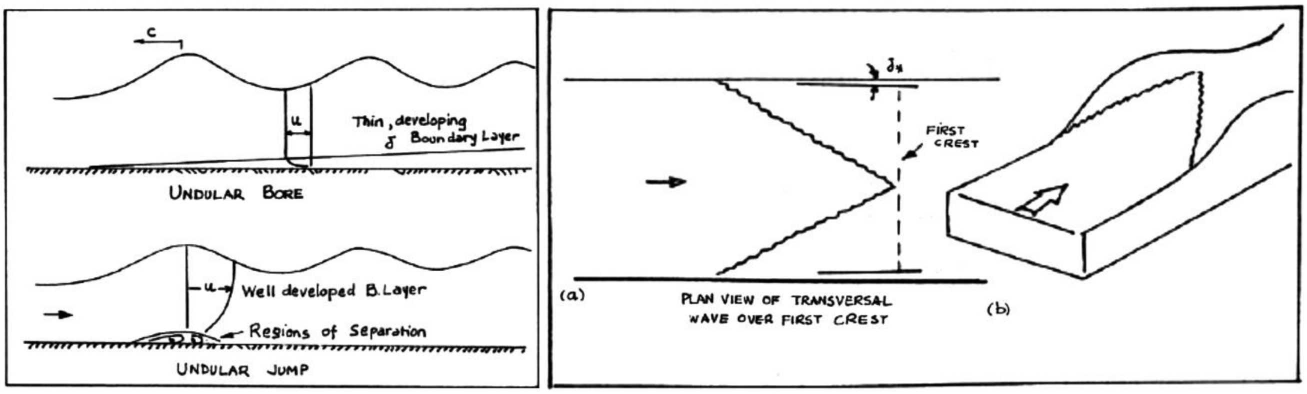


Figure 8. (Left) Comparison of boundary layer features in undular bore and undular jump (Montes 1986). (Right) Shock waves linked to development of wall boundary layers close to first crest (Montes 1986). As F_1 is increased above that suggested in the plot, a roller is formed near the first wave crest. A further increase of F_1 lengthens roller extension, until 'diamond pattern' of shock waves becomes trapped below roller and thus is hidden: the *undular jump* is transformed into a *direct hydraulic jump* (description of experiments by authors at VAW-ETH Zurich after a coffee talk in 2010)

The undular jump model developed by Montes (1986) is a generalization of Serre's (1953) equations using a Prandtl-type power law velocity profile of exponent *N*. The system of equations reads

$$\begin{aligned} &\frac{\mathrm{d}E}{\mathrm{d}x}=\sin\theta-\frac{U^2}{CR}, \\ &E=h\cos\theta+\frac{(1+N)^2}{N(N+2)}\frac{U^2}{2g}+\frac{(1+N)^2}{N(2+3N)}\frac{U^2}{g}\left[h\frac{\mathrm{d}^2h}{\mathrm{d}x^2}-\frac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\right]. \end{aligned} \qquad [4]$$

Here *U* is the depth-averaged velocity, θ the bottom inclination angle, *E* specific energy, *C* the Chezy resistance coefficient and *R* the hydraulic radius. Montes found that the integration of the system using shooting methods, as Milne's 4th-order predictor-corrector method, is by no means easy, given the sensitivity of the solution to the flow conditions upstream of the jump. This model was generalized by Montes and Chanson (1998) by inclusion of a detailed boundary-layer type model to compute the bed shear stress, whereas Chanson and Montes (1995) conducted a detailed set of experiments.

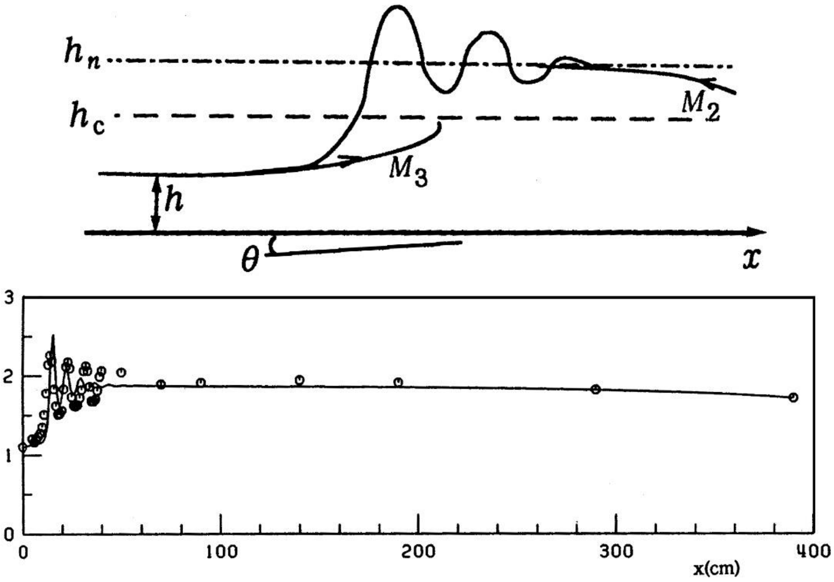


Figure 9. (Top) Sketch of undular jump formed at transition from M3 to M2 curves; (bottom) data for q = 57.9 cm^2/s, h_1 = 1.1 cm, sinθ = 1/400) (Hosoda and Tada 1994)

A second mayor contribution of this period is due to Hosoda and Tada (1994). Prof. Hosoda, from Kyoto University, Japan, is a former collaborator of Prof. Iwasa. Hosoda continued his work developing a Boussinesq-type momentum equation accounting for turbulence effects as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{h^2}{2}\cos\theta+hU^2+\frac{1}{3}h^2U^2\frac{\mathrm{d}^2h}{\mathrm{d}x^2}-\frac{1}{3}hU^2\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2\right]=gh\sin\theta-\frac{\tau_b}{\rho}+\frac{\mathrm{d}}{\mathrm{d}x}\left(D_m h\frac{\mathrm{d}U}{\mathrm{d}x}\right). \qquad [5]$$

Here, ρ is water density, D_m eddy viscosity and τ_b the bottom shear stress. Hosoda and Tada (1994) applied Eq. (5) to the undular jump formed at the transition from an M3 to an M2 curve (Fig. 9 top). Discretizing Eq. (5) based on the Kawamura higher-order upwind finite-difference scheme, and solving iteratively the governing implicit system of equations, the agreement between computations and their experiments is good (Fig. 9 bottom).

Grillhofer and Schneider (2003) developed an asymptotic solution of the RANS equations for near-critical flows at large Reynolds numbers and obtained an ordinary differential equation describing the solution for the undular jump profile.

Figure 10. Undular hydraulic jumps in VAW-ETH research studies (i) 2D (left), (ii) 3D (right)

Castro-Orgaz *et al*. (2015) developed a depth-averaged RANS model based on previous works by Montes (1986), Montes and Chanson (1998) and Hosoda and Tada (1994). They found reasonable agreement of all relevant flow features with observations for 2D undular jumps (Fig. 9 bottom, left, for F_1<1.2).This model is compared in Fig. 11 with 2D RANS solutions (Schneider *et al*. 2010) and experiments (Gotoh *et al*. 2005), showing excellent agreement. The main limitation was - and still is - that all models available so far are 1D, whereas the 3D flow structure of the undular jump (Fig. 10 right) for F_1>1.2 suggests that a 2D depth-averaged model is required. It clearly points that any further attempt to relate the undular jump to the undular surge for advancing in knowledge of the former is misleading: these are simply two notably different flows.

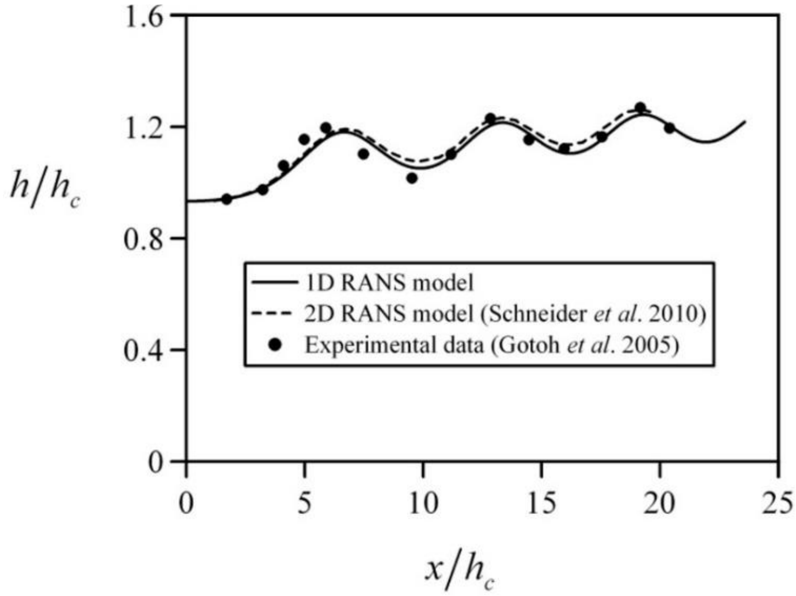


Figure 11. Comparison of undular jump solutions for F_1 = 1.11, with h_c as the critical depth (Castro-Orgaz and Hager 2017)

4. CONCLUSIONS

Both the undular hydraulic jump and the undular bore are beauties in the field of open channel hydraulics, given their shape and complexity, but they also count to the most complex features in this field. It can be stated that the extremely weak forms of these flows are by now explored, whereas flows with wave breaking, along with air entrainment and large turbulence generation, are still far away from the current knowledge. This review on the advances of the undular flows in the transcritical flow region would like to give insight into both the long way it took to understand the essences of the flow patterns, and also would like to indicate how one should proceed in the future to finally detect one of the last enigmas of open channel features, at least in the steady flow environment. Remember that Boussinesq started his outstanding researches nearly 150 years ago, resulting in the Boussinesq equations, which still serve as the foundation of the present computational models involving significant streamline curvature and inclination. His work has to be developed in the light of modern means to solve these equations with a rigorous approach, finally revealing the secrets of open channel flows with regard to engineering applications.

ACNOWLEDGEMENT

The Authors would like to dedicate this work to José Sergio Montes-Videla (J.S. Montes), a true pioneer in the study of the undular hydraulic jump. His 1986-"Congress" paper is an exemplary case on how groundbreaking ideas should not necessarily be searched by simply looking at papers published in journals of high impact factors!

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