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Computation of Gradually Varied Flow by Fourth Order Runge-Kutta Method (SRK)

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ABSTRACT

The main hydraulic engineering activities related to flow in open channels involve the calculation of gradually varied flow profiles (GVF) (Subramanya, 1990); the same one that requires a considerable effort in the analysis and solution of the problems under consideration, such as: flooding of land due to the construction of dams, determination of flow profiles due to the various hydraulic structures in the channel and estimation of flood areas. This paper outlines an efficient algorithm of numerical solution of the differential Gradually Varied Flow (GVF) based on the **Standard Fourth Order Runge-Kutta Method (SRK)** for permanent flow in prismatic channels without resorting to tables or traditional interpolation techniques. The results show that the calculation procedure, implemented in **MatLab**, meets a high level of convergence and a very high computational efficiency in terms of accuracy and speed.

Keywords: Gradually varied flow, flow profiles, numerical solution

1 INTRODUCTION

Gradually varied flow (GVF) is a steady non-uniform flow in a prismatic channel with gradual changes in its water-surface elevation. The backwater produced by a dam or weir across a river and the drawdown produced at a sudden drop in a channel are typical examples of GVF, where the velocity, water surface slope and energy slope varies along the channel.

The governing equation for steady gradually varied flow in open channels is a non-linear first order differential equation which can be integrated by analytical methods only under very restricted conditions. The importance of the phenomena of gradually varied flow in the both natural and man-made systems has led to the development of a number of approaches to approximate numerical solution of the equation. Those methods of computation described in open channel hydraulics books such as Chow (1952) and Henderson (1966) have been developed primarily for desk calculation. However, in order to integrate the gradually varied flow equation is important to develop an efficient method of integration for use with the computer.

2 THE GRADUALY VARIED FLOW MODEL

The two basic assumptions involved in the analysis of GVF are:

1. The pressure distribution at any section is assumed to be hydrostatic because the gradual changes in the surface curvature give rise to negligible normal accelerations.

2. The resistance to flow at any depth is assumed to be given by the corresponding uniform-flow equation, such as the Manning's formula, with the condition that the slope term to be used in the equation is the energy slope and not the bed slope, given by:

$$S_f = \frac{n^2 V^2}{R^{4/3}} \tag{1}$$

Differential equation of GVF

Consider the total energy H of a gradually-varied flow in a channel of small slope as:

$$H = Z + y + \alpha \frac{V^2}{2g} \tag{2}$$

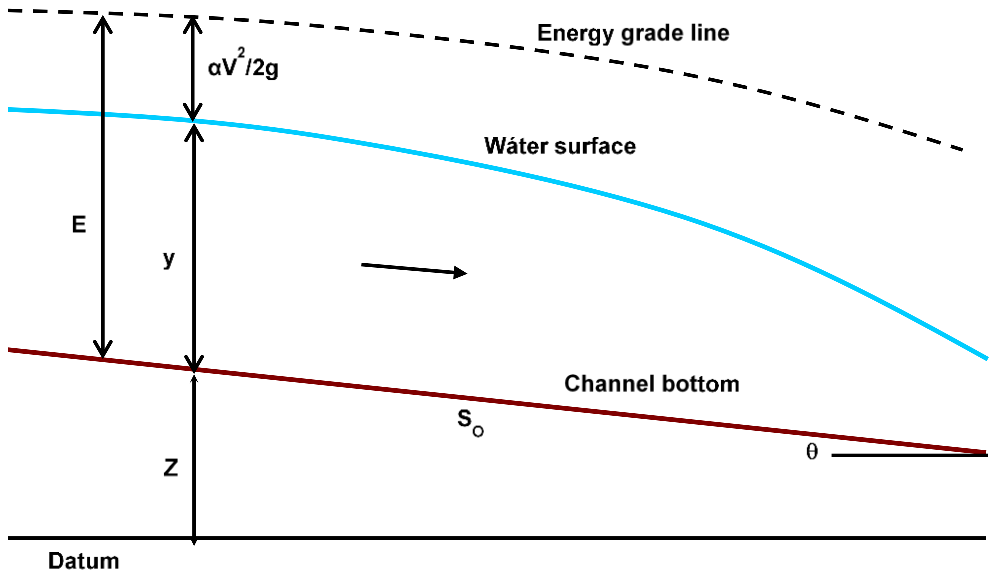

Figure 1: schematic sketch of a gradually-varied flow

Since the water Surface, in general, varies in the longitudinal (x) direction, the depth of flow and total energy are functions of x. Differentiating Eq. [2] with respect to x

$$\frac{dH}{dx} = \frac{dZ}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\alpha \frac{V^2}{2g}\right) \qquad [3]$$

In the equation (3), the meaning of each term is as follows:

1. $\frac{dH}{dx} = -S_f$ represents the energy slope
2. $\frac{dH}{dx} = -S_o$ represents the bottom slope
3. $\frac{dy}{dx}$ represents the water-surface slope relative to the bottom of the channel
4. $\frac{d}{dx}\left(\alpha \frac{V^2}{2g}\right) = \frac{d}{dy}\left(\alpha \frac{Q^2}{2gA^2}\right)\frac{dy}{dx} = -\alpha \frac{Q^2}{gA^3}\frac{dA}{dy}\frac{dy}{dx} = -\alpha \frac{Q^2}{gA^3} T \frac{dy}{dx}$ where: $\frac{dA}{dy} = T$

Eq. [3] can now be rewritten as: $-S_f = -S_o + \frac{dy}{dx} - \alpha \frac{Q^2 T}{gA^3}\frac{dy}{dx}$

Rearranging and considering α = 1.0 we obtain the basic differential equation of GVF and is also know as the dynamic equation of GVF:

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - \frac{Q^2 T}{gA^3}} \qquad [4]$$

Classification of flow profiles

For a given channel, the normal depth y_n and the critical depth y_c are fixed depths for a given flow Q, roughness coefficient n and slope S_o. It is possible to establish the following three relations between the normal and critical depths: $y_n > y_c$, $y_n < y_c$ e $y_n = y_c$. Then it should be indicated that there are two cases where y_n does not exist, when the channel bottom is horizontal (So = 0) and when the channel has an adverse or negative slope. Based on the above, the channels can be classified into five categories and for each of the five categories the lines corresponding to the normal depth (if it exist) and the critical depth can be schematized in the longitudinal profile. These lines divide the flow space into three regions, such as:

- Region 1: region above the line of greatest depth of water (normal or critical).
- Region 2: region between the two lines corresponding to the normal and critical line.
- Region 3: region between the bottom of the channel and the lowest depth line (normal or critical).

Tabla 1. Classification of Channels for the FGV Study

N°	Channel slope	Symbol	Condition	Remark
1	Mil slope	M	$y_n > y_c$	Subcritical flow at normal depth
2	Steep slope	S	$y_n < y_c$	Supercritical flow at normal depth
3	Critical slope	C	$y_n = y_c$	Critical flow at normal depth
4	Horizontal bed	H	$S_o = 0$	Cannot sustain uniform flow
5	Adverse slope	A	$S_o < 0$	Cannot sustain uniform flow

Depending on the channel category and the flow region, water surface profiles may have characteristic shapes. A FGV profile can present an increase or decrease in the depth of water in the flow direction depending if the term $\frac{dy}{dx}$ in Eq. [4] is positive or negative.

From Eq. [4] so that $\frac{dy}{dx}$ be positive you must meet the following conditions:

- The numerator > 0 and the denominator > 0: $K > Kn \wedge Z > Zc$ or $y > yn \wedge y > yc$
- The numerator < 0 and the denominator < 0: $K < Kn \wedge Z < Zc$ or $y < yn \wedge y < yc$

In a similar way so that $\frac{dy}{dx}$ be negative in Eq. [5], the following conditions must be met:

- the numerator is> 0 and the denominator <0: Kc > K > Kn or yc > y > yn
- the numerator is <0 and the denominator> 0: Kn > K > Kc or yn > y > yc

As an aid in the determination of flow profiles in different regions, it is important to analyze the behavior of $\frac{dy}{dx}$ in the Eq. [4]:

- When: $y \to y_n$, $\frac{dy}{dx} \to 0$, the surface of water approaches the normal depth asymptotically.
- When: $y \to y_c$, $\frac{dy}{dx} \to \infty$, the water surface reaches the critical depth line vertically. This information is important only as an indication of the trend of the flow profile. In fact, the formation of large curvatures in areas near the critical depth line violates the assumption of the nature of the FGV, so the calculation of the FGV profile must start or end at a very close distance from the normal depth line.
- When: $y \to \infty$, $\frac{dy}{dx} \to S_o$, the surface of water reaches a very large depth like a horizontal asymptote.

Based on this information, 12 types of gradually varied flow profiles are presented in the following table. The curvatures in region 1 have positive slopes; these are commonly known as backwater curves. Similarly, all the curves in region 2 have negative slopes and are known as drawdown curves. At the critical depth the FGV equation is not strictly applicable in this area very close to the critical depth.

Table 2. Classification of GVF profiles

Channel	Region	Condition	Type
Mild slope	1	$y > y_n > y_c$	M_1
	2	$y_n > y > y_c$	M_2
	3	$y_n > y_c > y$	M_3
Steep slope	1	$y > y_c > y_n$	S_1
	2	$y_c > y > y_n$	S_2
	3	$y_c > y_n > y$	S_3
Critical slope	1	$y > y_n = y_c$	C_1
	3	$y < y_n = y_c$	C_3
Horizontal bed	2	$y > y_c$	H_2
	3	$y < y_c$	H_3
Adverse slope	2	$y > y_c$	A_2
	3	$y < y_c$	A_3

3 NUMERICAL SOLUTION

Due to the practical importance of calculating GVF it has been an aspect of great interest since the last 150 years. Dupuit (1948) was perhaps the first to attempt to integrate the differential equation of the FGV. In recent periods efforts have been made to integrate the GVF equation through the use of simple flow resistance equations, such as the Chezy equation, and other simplifications in the channel geometry such as rectangular or parabolic. Subsequently Bakhmeteff developed a satisfactory method involving the use of varied flow functions to different types of channels; this method has suffered successive adjustments and refinements over time and finally Ven Te Chow (1955) proposed a comprehensive method using only the gradually varied flow function.

Simultaneously with the development of the direct integration method, for practical cases, various solution procedures comprising graphical and numerical methods evolved for use by engineers. The advent of high-speed computers has allowed the development of computer programs, using sophisticated numerical techniques, for the solution of the FGV equation in both prismatic channels and natural channels. These methods of calculating the FGV profiles can be classified as:

- Direct integration
- Graphic integration
- Numerical integration

Each of the indicated methods has a particular solution procedure for each type of problem to solve and it is not possible to generalize its application to solve all types of problems. It is also appropriate to indicate that the procedure for calculating FGV in an artificial channel may not be applicable to natural channels with irregular sections, so there are some specific methods to be applied to natural channels or rivers.

The basic differential equation of GVF, Eq. [4], can be expressed as:

$$\frac{dy}{dx} = F(y) \qquad [5]$$

In which $F(y) = \dfrac{S_o - S_f}{1 - \dfrac{Q^2T}{gA^3}}$ and is a function of y only for a given S_o, n, Q and channel geometry. For a particular solution to be calculated, one condition needs to be specified and this will normaly be the depth of flow corresponding to a specified value of x. A class of methods which is particularly suitable for direct solution of Eq. [5] when it is non-linear is that which is described as the Standard Fourth Order Runge – Kutta method (SRK), (Subramanya, 1982):

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad [6]$$

In which

$$k_1 = \Delta x.F(y_i)$$
$$k_2 = \Delta x.F\left(y_i + \frac{k_1}{2}\right)$$
$$k_3 = \Delta x.F\left(y_i + \frac{k_2}{2}\right)$$
$$k_4 = \Delta x.F(y_i + k_3)$$

For a prismatic trapezoidal channel:

$$F(y_i) = \frac{dy}{dx} = \frac{S_o - S_f}{1 - \frac{Q^2 T}{gA^3}} = \frac{S_o - S_{fi}}{1 - \frac{Q^2(b + 2zy_i)}{g(by_i + zy_i^2)^3}} \quad [7]$$

In which

$$S_{fi} = \frac{n^2 V^2}{R^{4/3}} = \frac{n^2 Q^2}{A^2 R^{4/3}} = \frac{n^2 Q^2}{(by_i + zy_i^2)^2 \left(\frac{by_i + zy_i^2}{b + 2y_i\sqrt{1 + z^2}}\right)^{4/3}}$$

$$k_i = \Delta x F(y_i) = \Delta x \frac{S_o - S_{fi}}{1 - \frac{Q^2(b + 2zy_i)}{g(by_i + zy_i^2)^3}}$$

4 RESULTS AND DISCUSSION

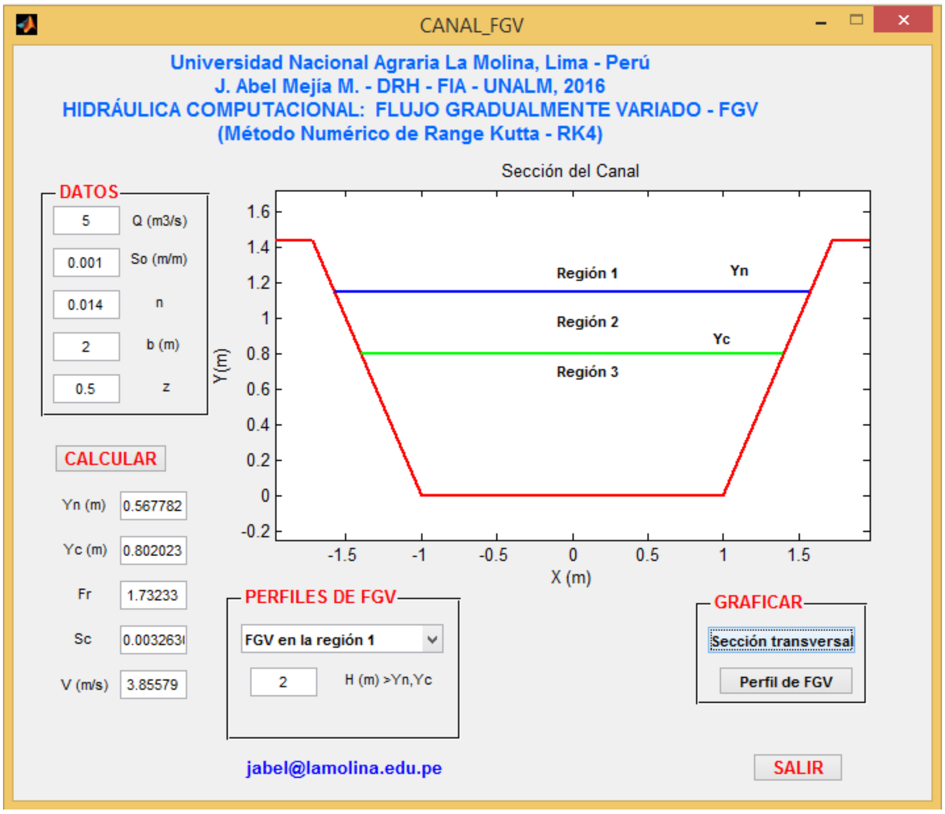

Figure 2. Subcritical flow and regions of flow profiles

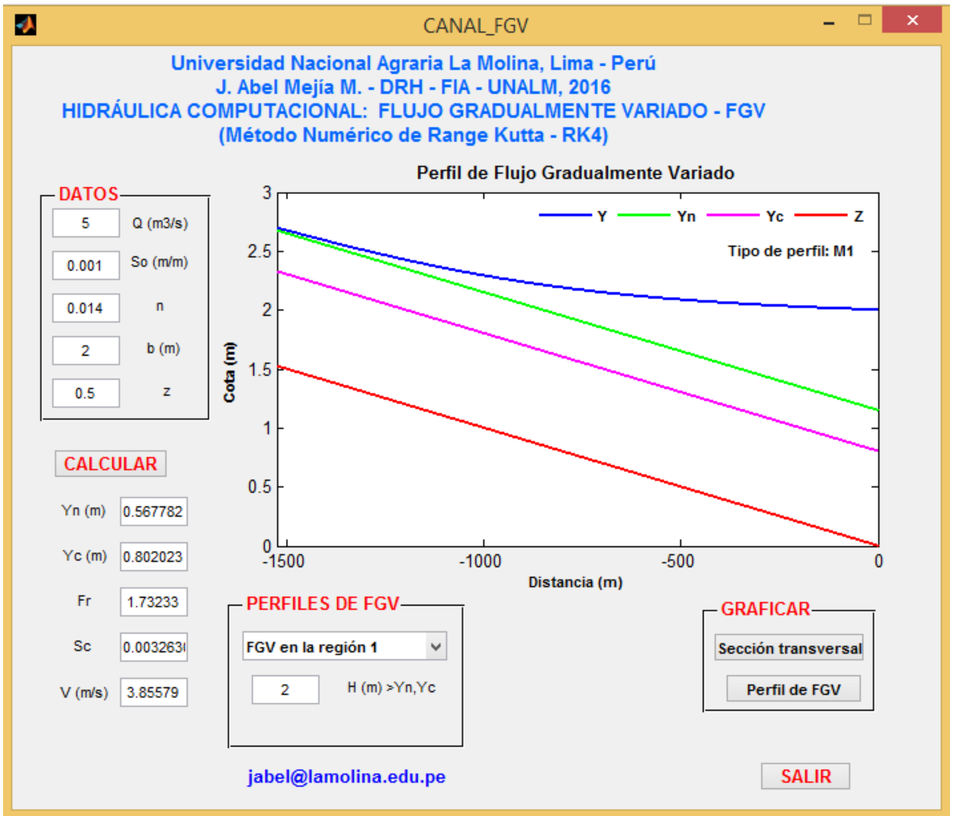

Figure 3. M1 type GVF profile in mild slope

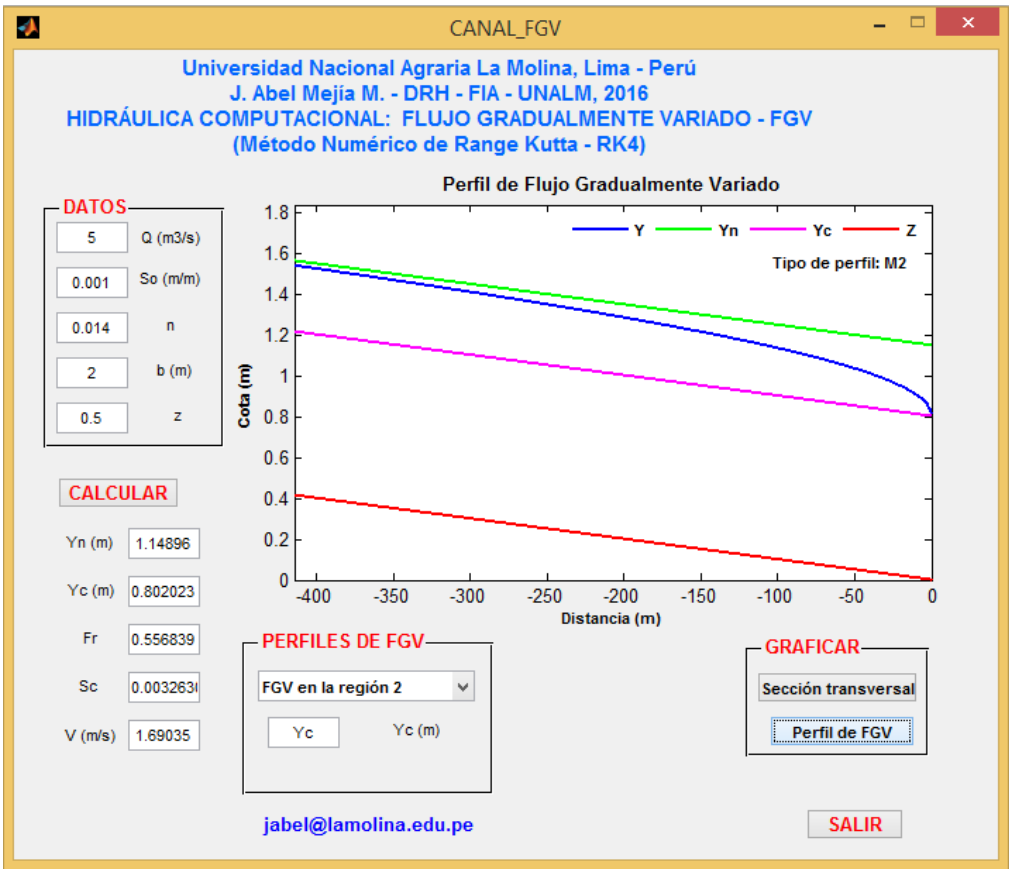

Figure 4. M2 type GVF profile in mild slope

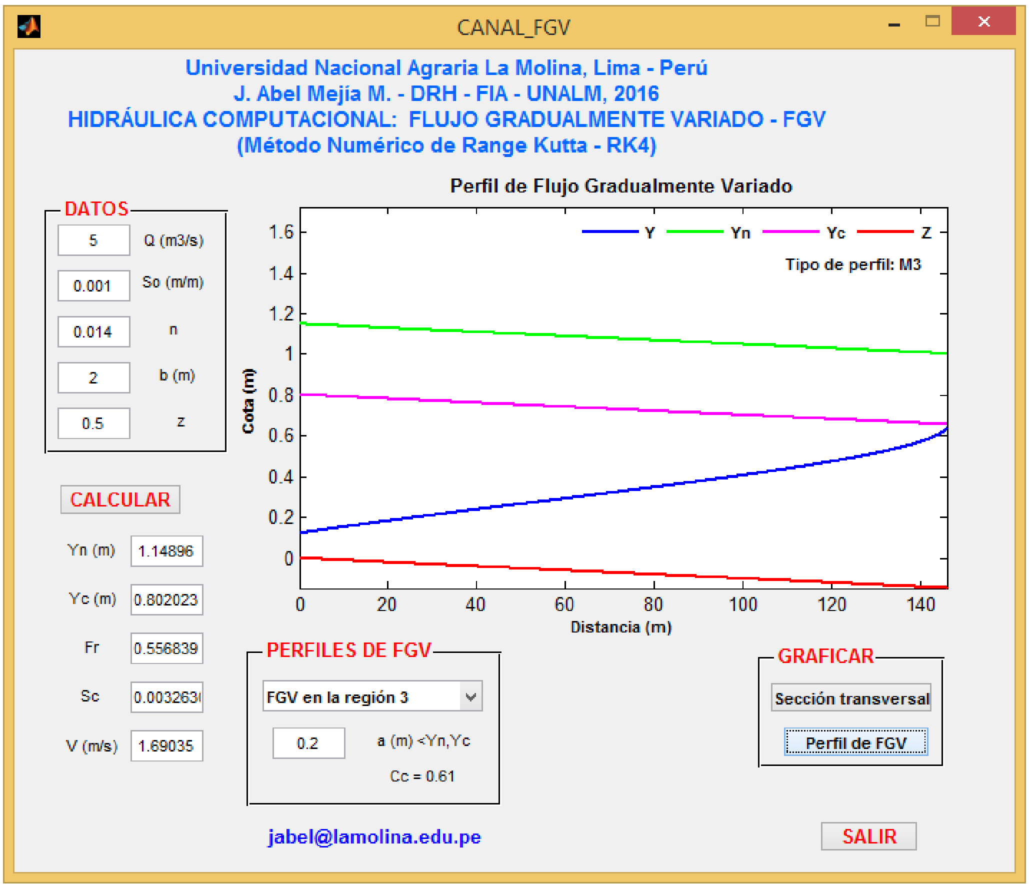

Figure 5. M3 type GVF profile in mild slope

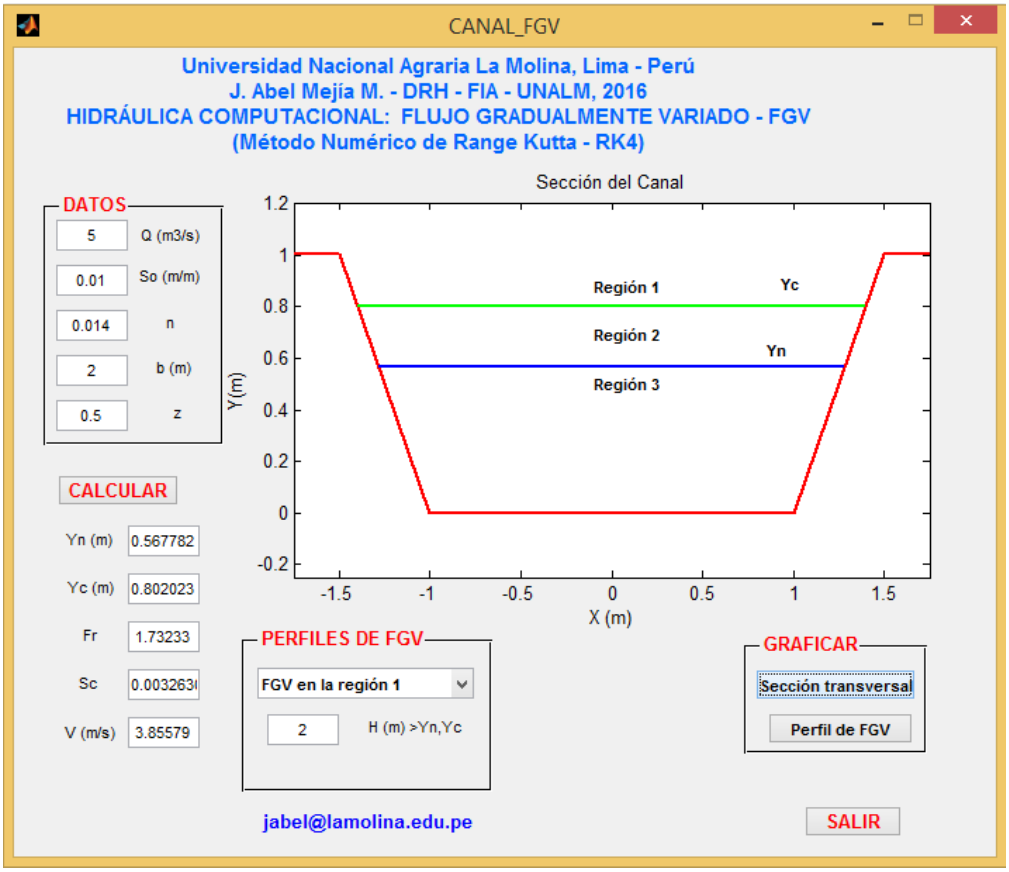

Figure 6. Supercritical flow and regions of flow profiles

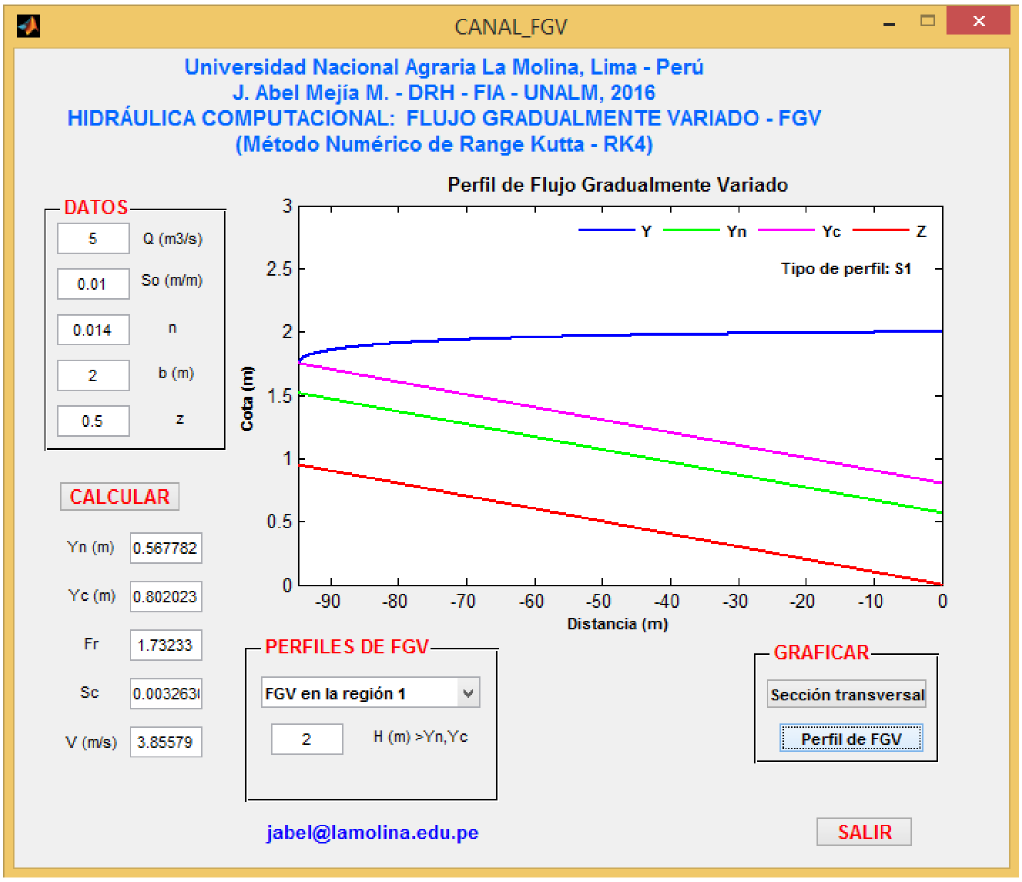

Figure 7. S1 type GVF profile in steep slope

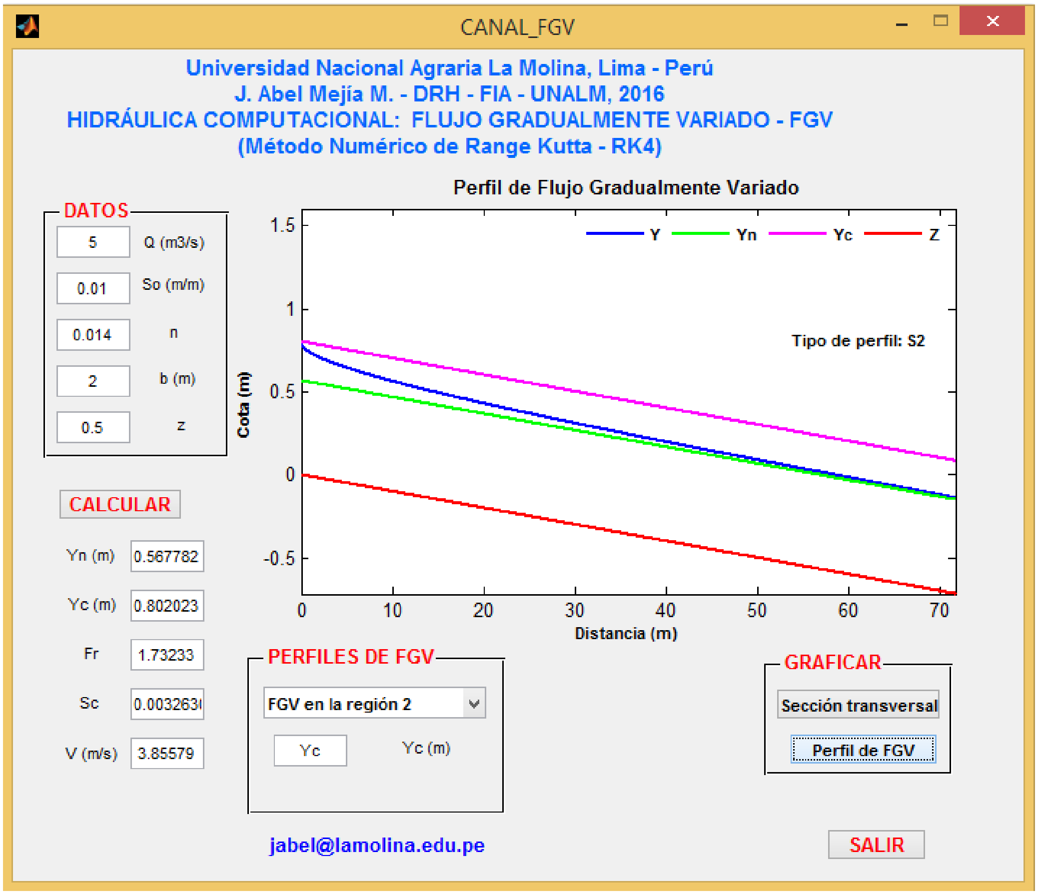

Figure 8. S2 type GVF profile in steep slope

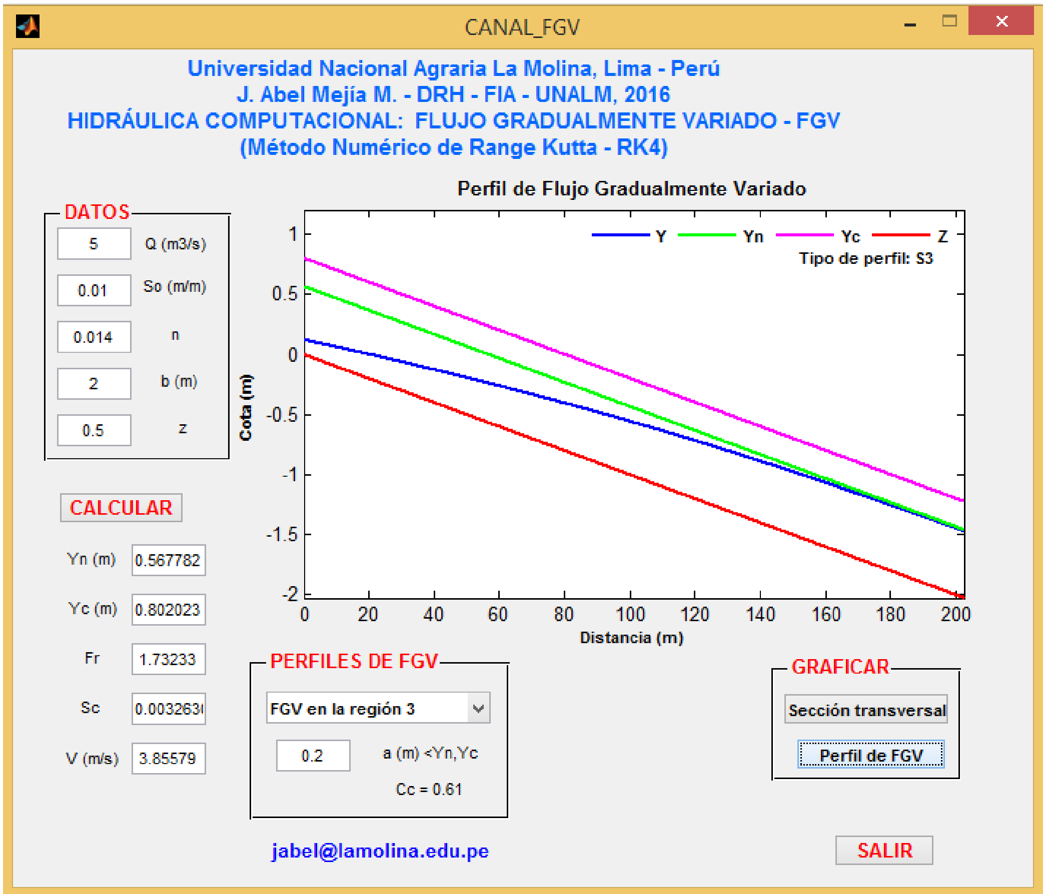

Figure 9. S3 type GVF profile in steep slope

About the presented flow profiles, we can indicate:

Mild Slope: The most common GVF profiles are the M1 type under subcritical flow conditions, figure 3. Flow obstructions such as dams, weirs, control structures and bottom elevations produce M1 backwater curves that extend several kilometers upstream before reaching the normal depth. The M2 profile occurs in sudden changes of slope of the channel where the flow changes from subcritical to supercritical as in hydraulic drops, figure 4. The curve of type M3 can occur in the flow leading from a spillway or a sluice gate to a mild slope. The beginning of the M3 curve is usually followed by a small contraction of rapidly-varied flow reaching the normal depth generally by the hydraulic jump, figure 5. In comparison to the profile M1, the profiles of type M2 and M3 are very short.

Steep Slope: The profile S1 is produced when the flow in supercritical regime is interrupted by an obstruction such as a dam a weir or a bottom elevation, figure 7. At the beginning of the curve, the flow changes from supercritical to subcritical flow through the hydraulic jump, extending downstream with a positive water-surface slope to reach a horizontal asymptote at the pool elevation. S2 type profiles occur at the entrance region of a steep channel and are generally short in length, figure 8. Free flow from a sluice gate in channels with a steep slope produce profiles of type S3 that start al the vena contracta until reaching the normal depth.

5 CONCLUSIONS

Runge – Kutta processes described above have been found to be very seccessful in all cases to which they have been applied. A fairly extensive set of computations has been carried out for integration of Eq. [4]. In summary, it has been established that integration can proceed either upstream or downstream and produce excellent results regardless of whether the flow is subcritical or supercritical. For a fixed interval of integration the SRK process achieved a specified accuracy of integration with alightly less computational expense.

REFERENCES

Apelt, C. J., Numerical Integration of the equation of gradually varied flow and spatially varied flow, Fourth Australian Conference on Hydraulics and Fluid Mechanics. Nov. 1971, pp. 146-153.

Chow, V. T., Open Channel Hydraulics, McGraw-Hill, New York, 1952

Henderson, F. M., Open Channel Flow, Macmillan, New York, 1966.

Subramanya, K. Flow in Open Channel, Tata MvGraw-Hill Publishing Company Limited, New Delhi, 1986